A homotopy-theoretic model of function extensionality in the effective topos (jww Daniil Frumin)

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Model structures on toposes

- There are two main classes of elementary toposes: Grothendieck and realizability toposes.
- Many examples of Grothendieck toposes carrying model structures.
- No (non-trivial) model structures on realizability toposes known.

Main obstacle

Realizability toposes are far from cocomplete. So we cannot appeal to the small object argument!

Today: a model structure on a full subcategory of the effective topos.

Assemblies

Write $\mathcal{P}_i(\mathbb{N})$ for the collection of inhabited subsets of the natural numbers.

An *assembly* is a pair (X, α) consisting of a set X together with a function $\alpha : X \to \mathcal{P}_i(\mathbb{N})$. A morphism of assemblies $(X, \alpha) \to (Y, \beta)$ is a function $f : X \to Y$ for which there is a partial recursive function φ such that:

> *For all $x \in X$ and $n \in \alpha(x)$, the function φ terminates on n and its value belongs to $\beta(f(x))$.*

We write Asm for the category of assemblies.

There is a functor $\nabla : \mathrm{Sets} \to \mathrm{Asm}$ obtained by sending X to (X, α) with $\alpha(x) = \{0\}$. This embeds the category of sets into the category of assemblies.

An assembly (X, α) is a *modest set* if $x = y$ whenever $\alpha(x) \cap \alpha(y)$ is inhabited. All the finite types are modest sets.

Effective topos (Hyland)

Given a regular category $\mathcal{C}$ its free exact completion is constructed as follows:

- Objects are pairs (X, R) where X is an object in $\mathcal{C}$ and $R \subseteq X \times X$ is an equivalence relation.
- Morphism $(X, R) \to (Y, S)$ are functional relations.

The resulting category is denoted by $\mathcal{C}_{ex/reg}$. There is an obvious embedding $\mathcal{C} \to \mathcal{C}_{ex/reg}$ obtained by mapping X to $(X, \Delta_X : X \to X \times X)$.

The *effective topos* is $\mathrm{Asm}_{ex/reg}$. As the name suggests, it is an elementary topos (with nno).

There is an embedding $\mathrm{Asm} \subseteq \mathrm{Eff}$ and hence an embedding $\nabla : \mathrm{Sets} \subseteq \mathrm{Eff}$; indeed, Sets is equivalent to the full subcategory of $\neg\neg$-sheaves on Eff, while Asm is equivalent to the full subcategory of $\neg\neg$-separated objects.

Earlier work by Jaap van Oosten

Theorem (Van Oosten)

One can define a finite limit preserving endofunctor P on Eff such that:

- Any object X in Eff becomes the objects of objects of an internal category with involution, with PX being the object of arrows.
- There a natural transformation $\Gamma : P \to P^2$ which, intuitively, contracts every path to the constant path at its starting point.

In other words, Eff carries the structure of a path object category in the sense of Van den Berg & Garner. This means that Eff carries a weak factorisation system in which the left maps are strong deformation retracts and the right maps are maps having a path lifting property. This yields a model of type theory with $\Pi, \Sigma, \mathbb{N}, 0, 1, \times, +, \mathrm{Id}$. But note that:

- This is not a model structure.
- Every object in Eff is fibrant.
- Function extensionality fails in this model (Jaap van Oosten).

Borrowing some ideas from Gambino and Sattler

Let us assume we work in an elementary topos $\mathcal{E}$ together with an interval.

Interval

An object I in $\mathcal{E}$ will be called an *interval* if it comes equipped with:

- maps $\delta_0, \delta_1 : 1 \to I$ such that $[\delta_0, \delta_1] : 1 + 1 \to I$ is monic, and
- connections $\wedge, \vee : I^2 \to I$.

For two maps $f : A \to B$ and $g : C \to D$ let us write $f \hat{\otimes} g$ (Leibniz product) for the canonical map

$$A \times D \cup_{A \times C} B \times C \to B \times D.$$

We will call a map a *fibration* if it has the right lifting property with respect to maps of the form $m \hat{\otimes} \delta_i$ where m is monic and $i \in \{0, 1\}$. An object X is *fibrant* if $X \to 1$ is a fibration; we will write $\mathcal{E}_f$ for the full subcategory on the fibrant objects,

A theorem yielding model structures

Theorem

Let $\mathcal{E}$ be an elementary topos and I be an interval as on the previous page. Then $\mathcal{E}_f$ carries a model structure in which the cofibrations are the monomorphisms, the fibrations are as on the previous page and the weak equivalences are homotopy equivalences.

Note:

- We do not claim to be able to construct a model structure on the whole of $\mathcal{E}$.
- If $\mathcal{E} = \mathrm{SSets}$ and $I = \Delta[1]$ then this yields the classical model structure on Kan simplicial sets.
- We are not assuming that $\mathcal{E}$ is cocomplete, so this theorem can also be applied to realizability toposes.

In particular, we can apply this theorem to Eff and $I = \nabla(2)$.

A model structure on a subcategory of Eff

The object $I = \nabla(2)$ is an interval in Eff. So we can define a map in the effective topos to be a fibration with respect to this object. Then:

Theorem

The category Eff_f carries a model structure in which the monomorphisms are the cofibrations, the fibrations are defined as before and the weak equivalence are the homotopy equivalences.

- We obtain a model of type theory with $\Pi, \Sigma, \mathbb{N}, 0, 1, +, \times$.
- In this model function extensionality does hold.
- Not every object in Eff is fibrant with respect to the current notion of fibrations. It is not hard to see that modest sets (so all finite types) and injective objects (like Ω) are fibrant, but in general showing that something is fibrant is rather difficult!
- If X is fibrant, then our path object X^I and Van Oosten's path object PX are homotopic; this means that when restricted to the fibrant objects our models are equivalent.

Restricting to assemblies

It is not so easy to understand this model. One reason is that most of the interesting homotopy-theoretic stuff must necessarily happen outside the assemblies.

Indeed, in the category of assemblies the map $[\delta_0, \delta_1] : 1 + 1 \to \nabla(2)$ is not just monic, but also epic. This means that any two paths in an assembly which share starting and end points must necessarily be identical! (So fibrant assemblies are hSets in a strong sense.)

This can be used to show:

Proposition

Any fibrant assembly is homotopic to a modest set. Therefore the homotopy category of Asm_f is equivalent to the category of modest sets.

Open questions

- Can one define a model structure on the whole of Eff?
- Are there examples of fibrant objects which are not hSets?
- Is there a suitable notion of fibrant replacement?
- Can the model be extended to include (univalent) universes?