

# Principal equivalence relations

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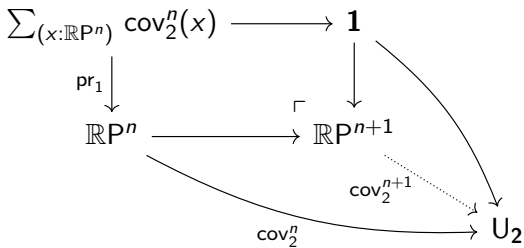
## **Part I: Classifying types (jww Ulrik Buchholtz)**

# Overview

- ▶ Definition of  $\mathbb{R}P^n$ .
- ▶ Definition of  $\mathbb{C}P^n$ .
- ▶ Principal H-spaces ('infinitely coherent groups').
- ▶ The general projective space construction.
- ▶ The principal H-space structure of loop spaces.
- ▶ Principal H-spaces are pointed connected types.

$$\mathbb{R}P : \mathbb{N}_{-1} \rightarrow \mathbf{U}$$

$$\text{cov}_2 : \prod_{(n:\mathbb{N}_{-1})} \mathbb{R}P^n \rightarrow \mathbf{U}_2$$



- ▶ For each  $n : \mathbb{N}$  there is an equivalence

$$\mathbb{S}^n \simeq \sum_{(x:\mathbb{R}P^n)} \text{cov}_2^n(x).$$

- ▶ We obtain the long exact sequence

$$\cdots \rightarrow \pi_{k+1}(\mathbb{R}P^n) \rightarrow \pi_k(\mathbf{2}) \rightarrow \pi_k(\mathbb{S}^n) \rightarrow \pi_k(\mathbb{R}P^n) \rightarrow \cdots$$

Since  $\pi_k(\mathbf{2}) = 0$  for  $k \geq 1$ , we get the isomorphisms

$$\pi_k(\mathbb{S}^n) = \pi_k(\mathbb{R}P^n)$$

for  $k \geq 2$ .

- ▶ The map

$$\text{cov}_2^\infty : \mathbb{R}P^\infty \rightarrow \mathbf{U}_2$$

is an equivalence.

$$\mathcal{O}_{\mathbb{S}^1} : U_{\mathbb{S}^1} \rightarrow U$$

such that  $\sum_{(A:U_{\mathbb{S}^1})} \mathcal{O}_{\mathbb{S}^1}(A) \times A$  is contractible

$$\mathbb{C}P : \mathbb{N}_{-1} \rightarrow U$$

$$\text{cov}_{\mathbb{S}^1} : \prod_{(n:\mathbb{N}_{-1})} \mathbb{C}P^n \rightarrow U_{\mathbb{S}^1}$$

$$\text{orient}_{\mathbb{S}^1} : \prod_{(n:\mathbb{N}_{-1})} \prod_{(x:\mathbb{C}P^n)} \mathcal{O}_{\mathbb{S}^1}(\text{cov}_{\mathbb{S}^1}^n(x))$$

$$\begin{array}{ccc}
 \sum_{(x:\mathbb{C}P^n)} \text{cov}_{\mathbb{S}^1}^n(x) & \longrightarrow & \mathbf{1} \\
 \text{pr}_1 \downarrow & \lrcorner & \downarrow \\
 \mathbb{C}P^n & \longrightarrow & \mathbb{C}P^{n+1} \\
 & \searrow & \swarrow \text{cov}_{\mathbb{S}^1}^{n+1} \\
 & & \sum_{(A:U_{\mathbb{S}^1})} \mathcal{O}_{\mathbb{S}^1}(A) \\
 \langle \text{cov}_{\mathbb{S}^1}^n, \text{orient}_{\mathbb{S}^1}^n \rangle & \longrightarrow & 
 \end{array}$$

- ▶ For each  $n : \mathbb{N}$  there is an equivalence

$$\mathbb{S}^{2n+1} \simeq \sum_{(x:\mathbb{C}P^n)} \text{cov}_{\mathbb{S}^1}^n(x).$$

- ▶ We obtain the long exact sequence

$$\cdots \rightarrow \pi_{k+1}(\mathbb{C}P^n) \rightarrow \pi_k(\mathbb{S}^1) \rightarrow \pi_k(\mathbb{S}^{2n+1}) \rightarrow \pi_k(\mathbb{C}P^n) \rightarrow \cdots$$

Since  $\pi_k(\mathbb{S}^1) = 0$  for  $k \geq 2$ , we get the isomorphisms

$$\pi_k(\mathbb{S}^{2n+1}) = \pi_k(\mathbb{C}P^n)$$

for  $k \geq 3$ .

- ▶ The map

$$\text{cov}_{\mathbb{S}^1}^\infty : \mathbb{C}P^\infty \rightarrow \sum_{(A:U_{\mathbb{S}^1})} \mathcal{O}_{\mathbb{S}^1}(A)$$

is an equivalence.

## Definition

A **principal H-space structure** on a type  $X$  with base point  $1_X$ , consists of

1. a type family  $\mathcal{O}_X : U_X \rightarrow U$  of **orientations**,
2. a **canonical orientation**  $o_X : \mathcal{O}_X(X)$ ,

such that the type

$$\sum_{(A:U_X)} \mathcal{O}_X(A) \times A$$

is contractible.

The **classifying type** of a principal H-space  $X$  is defined to be

$$\mathbf{B}X := \sum_{(A:U_X)} \mathcal{O}_X(A).$$

*This is a pointed connected type with loop space  $X$ .*



# The general projective space construction

$$P(X) : \mathbb{N}_{-1} \rightarrow U$$

$$\text{cov}_X : \prod_{(n:\mathbb{N}_{-1})} P(X)^n \rightarrow U_X$$

$$\text{orient}_X : \prod_{(n:\mathbb{N}_{-1})} \prod_{(x:P(X)^n)} \mathcal{O}_X(\text{cov}_X^n(x))$$

$$\begin{array}{ccc}
 \sum_{(x:P(X)^n)} \text{cov}_X^n(x) & \longrightarrow & \mathbf{1} \\
 \text{pr}_1 \downarrow & & \downarrow \\
 P(X)^n & \longrightarrow & P(X)^{n+1} \\
 & & \lrcorner \\
 & & \langle \text{cov}_X^{n+1}, \text{orient}_X^{n+1} \rangle \\
 & \searrow & \downarrow \\
 & & \sum_{(A:U_X)} \mathcal{O}_X(A) \\
 \langle \text{cov}_X^n, \text{orient}_X^n \rangle & \searrow & \\
 & & \downarrow \\
 & & \sum_{(A:U_X)} \mathcal{O}_X(A)
 \end{array}$$

- ▶ For each  $n \in \mathbb{N}$  there is an equivalence

$$X^{*(n+1)} \simeq \sum_{(x:P(X)^n)} \text{cov}_X^n(x).$$

- ▶ We obtain the long exact sequence

$$\cdots \rightarrow \pi_{k+1}(P(X)^n) \rightarrow \pi_k(X) \rightarrow \pi_k(X^{*(n+1)}) \rightarrow \pi_k(P(X)^n) \rightarrow \cdots$$

- ▶ The map

$$\text{cov}_X^\infty : P(X)^\infty \rightarrow \mathbf{BX}$$

is an equivalence.

## Lemma

Any loop space can be given the structure of a principal  $H$ -space.

Proof (Definition of the type of orientations).

Define  $P : X_{x_0} \rightarrow \text{Type}$  by  $P(x) := (x = x_0)$ . Then we have

$$\begin{array}{ccccc} \Omega(X) & \longrightarrow & \mathbf{1} & \longrightarrow & \tilde{U}_X \\ \downarrow & \lrcorner & \downarrow_{x_0} & \lrcorner & \downarrow \\ \mathbf{1} & \xrightarrow{x_0} & X_{x_0} & \xrightarrow{P} & U_X \end{array}$$

Now take

$$\begin{aligned} \mathcal{O}_X(A) &::= \text{fib}_P(A) \\ \mathcal{o}_X(\Omega(X)) &::= \langle x_0, \text{refl}_{\Omega(X)} \rangle \end{aligned}$$



## Proof of the contractibility property.

We have a commuting triangle

$$\begin{array}{ccc} X_{x_0} & \xrightarrow{\cong} & \sum_{(A:U_X)} \text{fib}_P(A) \\ & \searrow P & \swarrow \text{pr}_1 \\ & U_X & \end{array}$$

Therefore, the following are equivalent

- ▶  $\sum_{(A:U_X)} \mathcal{O}_X(A) \times A$  is contractible
- ▶  $\sum_{(x:X_{x_0})} P(x)$  is contractible

The latter is obvious. □

## Theorem

*The map  $\mathbf{B}$  from principal  $H$ -spaces to the type of (small) pointed connected types is an equivalence.*

## **Part II: Principal equivalence relations**

# Overview

- ▶ Principal equivalence relations ('infinitely coherent type-valued equivalence relations').
- ▶ The quotient approximation construction.
- ▶ The (modified) join construction.

## Definition

A **principal equivalence relation** on a type  $A$  consists of

- ▶ A binary relation  $R : A \rightarrow (A \rightarrow U)$  with a proof  $\rho : \prod_{(a:A)} R(a, a)$  of reflexivity,
- ▶ A type family

$$\mathcal{O}_R : \text{im}(R) \rightarrow U$$

of  **$R$ -orientations** on the predicates  $P : A \rightarrow U$  in the image of  $R$ , with a **canonical  $R$ -orientation**

$$\mathcal{O}_R : \prod_{(a:A)} \mathcal{O}_R(R(a)),$$

such that the type

$$\sum_{(P:\text{im}(R))} \mathcal{O}_R(P) \times P(a)$$

is contractible for every  $a : A$ .



- ▶ A principal equivalence relation on  $\mathbf{1}$  is the same thing as a principal H-space.

### Example

Let  $A$  be a type, and let  $\circ$  be a modality. Define

$$R(a, b) := \circ(a = b)$$

Then the type

$$\sum_{(P:\text{im}(R))} P(a)$$

is contractible for any  $a : A$ . So we may take  $\mathcal{O}_R(P) := \mathbf{1}$ .

## Definition

Given a principal equivalence relation  $R$  on  $A$ , we define the **quotient**

$$A/R \equiv \sum_{(P:\text{im}(R))} \mathcal{O}_R(P).$$

We define the **quotient map**  $q_R : A \rightarrow A/R$  by

$$q_R(a) \equiv \langle q'_R(a), o_R(a) \rangle.$$

## Lemma

*Given a principal equivalence relation  $R$  on  $A$ , we have by the encode-decode method an equivalence*

$$(q_R(a) =_{A/R} q_R(b)) \simeq R(a, b)$$

*for any  $a, b : A$ .*

$$\begin{array}{ccc}
 \sum_{(a:A)} \sum_{(x:X)} Y(x, a) & \xrightarrow{\pi_2} & X \\
 \pi_1 \downarrow & \lrcorner & \downarrow \\
 A & \longrightarrow & A +_Y X \\
 & \searrow & \downarrow \langle Y', o_{Y'} \rangle \\
 & & \sum_{(P:\text{im}(R))} \mathcal{O}_R(P) \\
 & \searrow & \uparrow \langle Y, o_Y \rangle \\
 & & X
 \end{array}$$

This defines an endomorphism on the type

$$\sum_{(X:U)} \sum_{(Y:X \rightarrow \text{im}(R))} \prod_{(x:X)} \mathcal{O}_R(Y(x)).$$

Via this endomorphism, we define the **quotient approximation sequence**

$$\begin{array}{ccccccc} [A/R]_0 & \xrightarrow{\text{inr}} & [A/R]_1 & \xrightarrow{\text{inr}} & [A/R]_2 & \xrightarrow{\text{inr}} & \dots \\ & \searrow [q_R]_0 & \downarrow [q_R]_1 & \swarrow [q_R]_2 & \swarrow [q_R]_3 & & \\ & & \sum_{(P:\text{im}(R))} \mathcal{O}_R(P) & & & & \end{array}$$

starting at  $[A/R]_0 \equiv A$  (or at  $[A/R]_{-1} \equiv \mathbf{0}$ ).

- ▶ Define  $[I_R]_n \equiv \text{pr}_1 \circ [q_R]_n : [A/R]_n \rightarrow A \rightarrow U$
- ▶ For each  $n \in \mathbb{N}$  and each  $b \in A$ , there is an equivalence

$$\sum_{(x:[A/R]_n)} [I_R]_n(x, a) \simeq \left( \sum_{(a:A)} R(a, b) \right)^{*(n+1)}.$$

- ▶ The map

$$[q_R]_\infty : [A/R]_\infty \rightarrow \sum_{(P:\text{im}(R))} \mathcal{O}_R(P)$$

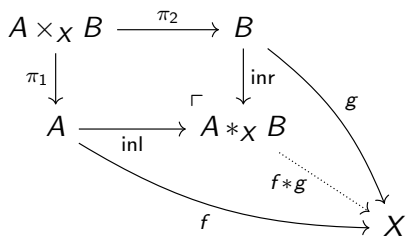
is an equivalence.

We obtain a map  $Q_A$  from the type of **principal equivalence relations** on  $A$  to the type of **surjective maps** out of  $A$ , with small codomains.

### Theorem

*For each type  $A$ , the map  $Q_A$  is an equivalence.*

# The join of maps



## Definition

For any  $f : A \rightarrow X$ , we define a sequence

$$\begin{array}{ccccccc} A_0 & \xrightarrow{i_0} & A_1 & \xrightarrow{i_1} & A_2 & \xrightarrow{i_2} & \dots \\ & \searrow f^{*0} & \downarrow f^{*1} & \nearrow f^{*2} & \nearrow f^{*3} & & \\ & & X & & & & \end{array}$$

The function  $f^{*n}$  is called the  $n$ -th join-power of  $f$ .

## Construction.

We take  $A_0 \equiv \mathbf{0}$ , with the unique map into  $X$ . Then we define  $A_{n+1} \equiv A_n *_X A$ , and  $f^{*(n+1)} \equiv f^{*n} * f$ . □



- ▶ In the real projective case:
  - ▶  $\mathbb{R}P^{n+1} = \mathbb{R}P^n *_{\mathbb{R}P^\infty} \mathbf{1}$ , and
  - ▶  $\text{cov}_2^n = (\text{cov}_2^0)^{*(n+1)}$
- ▶ In the complex projective case:
  - ▶  $\mathbb{C}P^{n+1} = \mathbb{C}P^n *_{\mathbb{C}P^\infty} \mathbf{1}$ , and
  - ▶  $\text{cov}_{\mathbb{S}^1}^n = (\text{cov}_{\mathbb{S}^1}^0)^{*(n+1)}$
- ▶ In the general projective case:
  - ▶  $P(X)^{n+1} = P(X)^n *_{\mathbf{B}X} \mathbf{1}$ , and
  - ▶  $\text{cov}_X^n = (\text{cov}_X^0)^{*(n+1)}$

## Theorem

The sequential colimit  $f^{*\infty}$  is an embedding, and has the universal property of the image inclusion

$$A \rightarrow \text{im}(f) \rightarrow X$$

## Corollary

The sequential colimit of the type sequence

$$\mathbf{0} \longrightarrow A \xrightarrow{\text{inr}} A * A \xrightarrow{\text{inr}} A * (A * A) \xrightarrow{\text{inr}} \dots$$

is the propositional truncation  $\|A\|_{-1}$ .

## Definition

A type  $X$ , which may itself be large, is said to be **locally small** if for all  $x, y : X$ , there is a type  $x =' y : U$  and an equivalence of type

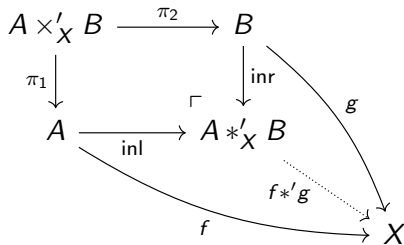
$$(x = y) \simeq (x =' y).$$

Examples:

- ▶ all types in  $U$ ,
- ▶ the universe  $U$ ,
- ▶ mere propositions of any size,
- ▶ for any  $A : U$  and any locally small type  $X$ , the type  $A \rightarrow X$ .

## The modified join of maps into locally small $X$

Take  $A \times'_X B \equiv \sum_{(a:A)} \sum_{(b:B)} f(a) ='_ g(b)$ .



## Theorem

*Assumptions:*

- ▶  $U$  is a univalent universe closed under pushouts,
- ▶ let  $X$  be a locally small type,
- ▶ let  $f : A \rightarrow X$  with  $A : U$ ,

Then we can construct a type  $\text{im}'(f) : U$ , a surjective map  $q'_f : A \rightarrow \text{im}'(f)$ , and an embedding  $i'_f : \text{im}'(f) \rightarrow X$  such that the triangle

$$\begin{array}{ccc} A & \xrightarrow{q'_f} & \text{im}'(f) \\ & \searrow f & \downarrow i'_f \\ & & X \end{array}$$

commutes, with the universal property of the image inclusion of  $f$ .

<i>level</i>	<i>equivalence structure</i>	<i>quotient operation</i>
-2	trivial relation	propositional truncation
-1	Prop-valued equivalence relation	set-quotient
0	pre-1-groupoid structure	Rezk completion
$\vdots$	$\vdots$	$\vdots$
$\infty$	'pre- $\infty$ -groupoid structure'	$\infty$ -quotient

- ▶ For any modality  $\circ$ , the modality of  $\circ$ -separated types.
- ▶  $n$ -truncation in any univalent universe closed under pushouts.
- ▶  $\mathbb{R}P^n, \mathbb{C}P^n, \dots$