This talk

I will survey some works connected to the formulation of a constructive model of univalence

Symmetric cubical sets

Symmetric cubical sets with connections

Uniformity condition

Why introduing connections?

Formalisation and consequences

Canonicity

This talk

Strategy we followed: to find a model of type theory with univalence in a constructive framework

Once this is done, the formulation of an associated type theory, with its associated operational semantics, is direct

The operational semantics corresponds in writing precisely the constructive semantics

Simplicial set model

$\langle\!\langle$Spaces$\rangle\!\rangle$ as Kan simplicial sets

Does not work constructively

If we try to prove e.g. that B^A is Kan if B is Kan, we see informally that we need to decide if a simplex of A is degenerate or not

Similarly if $E \to B$ Kan fibration and we try to build an equivalence $E(b_0) \to E(b_1)$ lifting a path $b_0 \to b_1$ in B

This informal remark can be transformed to a Kripke counter-model (j.w.w. Marc Bezem)

Simplicial set model

For B^A see the Ph.D. thesis of Erik Parmann (Bergen, February 2016)

This counter-example holds even if the Kan filling condition is expressed as a given operation (and not simply as an existential statement)

Simplicial set model

So we need to find a refinement of Kan's definition

First natural attempt: the restriction map $B^{\Delta^n} \rightarrow B^{\Lambda^n_k}$ has a section

and the corresponding refinement for fibration

This is enough for the result about B^A being Kan

However this does not seem sufficient for interpreting *dependent product*?

Simplicial set model

In the description of the model structure on simplicial sets the interpretation of dependent product reduces to the fact that trivial cofibrations are stable under pullbacks along Kan fibrations

In turn this reduces to the fact that the model structure is *proper*

But this is a non trivial fact, which is proved using minimal fibrations (in A. Joyal's presentation)

Goerss and Jardine uses topological spaces at this point

Partial elements

Instead the refinement is the following for the definition of being contractible

In an elementary topos, let $\mathsf{part}(A)$ be the object of partial element of A

We have a canonical map $A \to \mathsf{part}(A)$

A is *contractible* iff this map has a retraction: we can extend any partial element to a total element

For simplicial set, this definition is *classically* equivalent to the usual definition, which is classically equivalent to being *injective*

N. Gambino and Ch. Sattler *Uniform fibrations and the Frobenius condition*

Partial elements

In the 《internal logic》 of the presheaf model

$\Gamma \vdash A$, not necessarily fibrant, is a family of sets $A\gamma,\ \gamma \in \Gamma$

For $\Gamma \vdash A$ to correspond to a Kan fibration

If we have $\gamma \in \Gamma^{\mathbb{I}}$ and $a_0 \in A\gamma_0$ and a *partial* section $a(i) \in A\gamma(i),\ i \in \mathbb{I}$ with $a(0) = a_0$ then can extend this to a *total* section $\tilde{a}(i) \in A\gamma(i)$

In the special case where the partial section is nowhere defined this means that we have the path lifting property

We only have to consider $\Gamma^{\mathbb{I}}$ and not all $\Gamma^{\mathbb{I}^n}$

Partial elements

How far can one go with an internal formulation in an elementary topos?

See work of Andy Pitts and Ian Orton

Axioms for Modelling Cubical Type Theory in a Topos, CSL 2016

Partially formalized in Agda

Cubical sets

The definition was suggested by a «nominal» definition of symmetric cubical set

Presheaf over the base category: objects are finite sets (of names/symbols) and an element of $Hom(I, J)$ is a map of domain J the elements of J to «disjoint» elements in $I \cup \{0, 1\}$ We think of the names as names for «direction»; not having an order on the name is very natural (homogeneity of the directions)

If the Kan filling operation is given by an explicit operation, it is natural to ask that this operation commutes with renaming and addition of new names

With this added uniformity condition one can interpret dependent products in a constructive framework

Cubical sets

Implemented in Haskell (December 2013)

There is a natural definition of a universe: $U(I)$ collection of all small types $I \vdash A$

There is a natural candidate for the Kan filling operation on the universe but it is not at all trivial to check that this works (done independently by G. Gonthier and S. Huber, December 2013)

Both for dependent product and universe, one has *first* to define Kan composition and then the Kan filling operation

In both cases, the definition of composition is much easier than the definition of filling

Cubical sets

There is also a natural way to transform a given equivalence

$f : A \to B$

to a path E between A and B

A hypercube $a \to b$ in E is defined by a cube a in A together with a cube (of the next dimension) $f\ a \to b$ in B

It is rather direct to check that the cubical set defined in this way has a Kan filling operation, provided A and B have one and f is an equivalence

Cubical sets

There is a natural interpretation of the circle as the Kan cubical set generated by $\mathsf{base},\ \mathsf{loop}(i)$ and Kan filling operation, taken as a *constructor*

When we tried to implement the elimination operation we discovered a problem with *linearity*, given $a : A,\ l : a \to a$

$f : S^1 \to A$

$f\ \mathsf{loop}(i) = l\ i$

It may be that i occurs in l!

So one has to take away the linearity condition

Cubical sets

The other problem was that we only interpret the computation rule for identity elimination as a *propositional* equality

Cubical sets with connections

To solve both problems, I tried to work with symmetric cubical sets *with diagonals* (and reversal) and connections, i.e. works with maps $J \rightarrow \mathsf{dM}(I)$

In a first attempt, we had an extra *regularity* condition (terminology from Hurewicz) on fibrations: the transport along a constant path of types is the identity function

This takes exactly care of the computation rule of identity elimination

Also one can then reduce filling to composition (and it was first thought that regularity was necessary)

The composition for universe has to be regular/normal

Cubical sets with connections

We can still transform an equivalence to a path

This suggested the glueing operation Glue $[\varphi \mapsto (T, w)]\ B$

We give a hypercube B is the universe and a collection of compatible equivalences of codomain some faces of B

Such a generalization comes naturally because of the uniformity condition with connections

Cubical sets with connections

Dan Licata (with Bob Harper, Carlo Angiuli and Ed Morehouse) found a problem with regularity for universe (May 30 2015)

This suggests strongly that we *cannot* have Path = Id

This suggested to look more carefully at the reduction of filling to composition: regularity is actually not needed for this

This simplified significantly the model

Composition for the Glueing could be defined almost as before

Cubical sets with connections

S. Huber noticed that we can also take away the regularity condition $B = \mathsf{Glue}\ []\ B$ for the glueing operation

Together with A. Mörtberg, they noticed further that we can derive univalence only using the glueing operation (which was only there a priori to transform an equivalence to a path)

Cubical sets with connections

In this version (June 2015), we have Path $\neq$ Id

Andrew Swan found a way to define Id in the first cubical set model (without connections), using a small object argument

His definition can be adapted for the model with connections, and can then be formulated in a 《closed form》: an element $a_0 \rightarrow a_1$ of the Id type is a hypercube where we mark explicitely what faces are constant

Cubical sets with connections

N. Gambino and P. Lumsdaine noticed the similarity of the glueing operation and some diagram in the proof of Theorem 3.4.1 in the paper presenting the simplicial model

In discussions with S. Huber, we noticed that the glueing operation can be seen as a part of the expression that the type $(X : U) \times \mathsf{Equiv}\ A\ X$ is contractible, by stating that any partial element can be extended to a total element

As it turned out, this is an equivalent formulation of univalence (which is also used at the beginning of the proof of Theorem 3.4.1)

Glueing

So an operation like glueing has to be present in any model of univalence

Since any path in the universe defines an equivalence, it follows from glueing that the universe has a composition operation

The proof is much simpler than for the model without connections

Cubical set model

We have a model of the univalence axiom in a constructive meta theory

Can we derive a model where

(1) Markov principle is not valid, or

(2) some continuity principle holds?

One proposal for doing this is to notice that the models can be expressed in CZF or NuPrl and then to work with models of these theories satisfying (1) or (2)

Glueing

It is very important to have a formal verification of this model

Mark Bickford has succeded in formalizing in NuPrl the glueing operation (May 2016)

This is a non trivial result, and it will be important to also have a formalization in intensional type theory with univalence and to compare the two formalizations

Higher Inductive Types

Since we have diagonals we can interpret the spheres without problems

The model justifies computation rules both for points and paths

We also have a model of suspension and propositional truncation

As far as I know, this is the first model of these operations (we solve the coherence problems indicated in the work of M. Schulman and P. Lumsdaine)

Formalization in NuPrl

It follows from this formalization that intensional type theory with univalence is not proof theoretically stronger than NuPrl

It follows that univalence does not add any proof theoretic strength

Cofibrations

Christian Sattler was able to define a (proper) Quillen model structure in a constructive framework

(This is very surprising since the duality between fibration/cofibration seems at first strongly linked to classical logic)

This uses in a crucial way the *cubical* structure

Is there such a constructive model structure for simplicial sets?

In this model, *cofibrations* are only special kind of monos

Cofibrations

An argument of Andrew Swan shows that we cannot have cofibrations $=$ monos in a constructive way and have the glueing operation

Take Y constant presheaf $\{\{0\},\{0|p\}\}$ and X constant presheaf $\{0\}$ where 0 is the *empty set*

If p holds we have an equivalence $X \rightarrow Y$

If we have a glueing operation and monos are cofibrations we should have a presheaf Z such that $Z = X$ if p holds and an equivalence $Z \rightarrow Y$

We then have $\forall x \in Z[0] \;\; x = 0$ iff p holds

So p is stable (implied by its double negation)

Canonicity

S. Huber has just finished the proof of *canonicity* (any closed term of type nat is convertible to a numeral) for the associated cubical type theory, using a presheaf version of Tait's computability method (1967)

There also we need to have a formal version of the proof

Canonicity

One defines computability predicate $I \Vdash t : A$ and relation $I \Vdash t = u : A$

E.g. $I \Vdash t :$ bool iff $J \vdash tf \rightarrow^* 0 :$ bool or $J \vdash tf \rightarrow^* 1 :$ bool for all $f : J \rightarrow I$

Reformulation of the 《expansion Lemma》 (similar, but independent of the work of Bob Harper and Carlo Angiuli)

If we have $I \Vdash A$ and $I \vdash t : A$ and we have a family of terms $J \Vdash u_f : Af$ with the conditions $J \vdash tf \rightarrow^* u_f : Af$ and $K \Vdash u_f g = u_{fg} : Afg$ for $f : J \rightarrow I$ and $g : K \rightarrow J$ then $I \Vdash t : A$ and $I \Vdash t = u_1 : A$

Canonicity

This suggests the following conjecture (weakening of Voevodsky's conjecture)

If we have $\vdash t : \mathsf{bool}$ in type theory extended with the univalence axiom *and* $\vdash t = 0 : \mathsf{bool}$ in cubical type theory can we show that the semantics of t is equal to 0 in the simplicial set model?

It should be enough to validate (classically) the rules of cubical type theory in the simplicial set model

Canonicity

A variation on propositional resizing would be to introduce $\Omega : U$ with rules

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash p : \mathsf{isProp}\ A}{\Gamma \vdash (A, p) : \Omega} \qquad\qquad \frac{\Gamma \vdash X : \Omega}{\Gamma \vdash X.1 \qquad \Gamma \vdash X.2 : \mathsf{isProp}\ (X.1)}$$

We conjecture that canonicity still holds for this extension