Homotopy Type Theory as a Foundation of Mathematics ?

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- Is HoTT only good for *synthetic homotopy theory*?
- Emphasize HoTT as an alternative to set theory.
- Shouldn't we use HoTT when doing the meta-theory of HoTT?
- Can use set-level HoTT to capture higher dimensional aspects of HoTT without presupposing them.

Platonism

- Platonism here means Mathematical Realism.
- Treat mathematical objects as if they were real objects.
- Sometimes reasonable abuse of language.
- Mathematical objects are mental constructions, and it is essential that we are aware of this.
- We can communicate our understanding based on shared intuitions.
- No universality of mathematical concepts.
- Rather convergent evolution.
  Aliens will know natural numbers.
  Wings have been invented several times.

Constructivism

- Cannot accept the platonic notion of absolute truth.
- Instead explain the meaning of a proposition by explaining what is evidence for it.
- In Type Theory: proposition as types.
- We should also reject the idea of sets as collections of preexisting objects.
- Type Theory: objects are constructed as elements of a type, cannot be viewed in isolation.
- Types are not collections!
- Membership of a type is a judgement, not a proposition.

Extensionality

- Since we cannot talk about objects in isolation, we cannot talk about their intensional properties.
- For this reason operations like $\cup$, $\cap$, $\subseteq$ on types have no role in Type Theory.
- The inability to talk about intensional properties gives rise to univalence.
- **intensional** Objects are understood by their way they are constructed.

  extensional Objects are understood by what we can say about them.
- Different to the distinction for real world semantics.

Two Dichotomies

- Propositions as truth vs evidence.
- Mathematical objects are organized as collections vs types.

	collections	types
truth	classical set theory	classical type theory
evidence	constructive set theory	constructive type theory

Propositions as types

- Doesn't mean that all types are propositions.
- HoTT: propositions are proof-irrelevant types.
- Refined translation:

$$P \vee Q \equiv ||P + Q||$$
$$\exists x : A.P(x) \equiv ||\Sigma x : A.P(x)||$$

- Types are more expressible than logic.
- E.g. can state

  Church's thesis *All functions are computable*

  Brouwer's continuity *All functions are continous*

  without unintended contradictions.
- Answers the question about the role of the axiom of choice in constructive mathematics.

Intensional Type Theory (ITT)

- Types have elements but no further structure.
- Equality type identifies objects which are constructed the same way.
- It reflects definitional equality but it allows hypothetical equalities.
- Equality type is inductively defined with only constructor **refl**.
- This view both justifies J and K (and hence uniqueness of equality proofs).
- Restriction to J is a historic accident.

Structural incompleteness of ITT

- ITT doesn't identify extensionally equal functions like $\lambda x.x + 0$ and $\lambda x.0 + x$.
- On the other hand it doesn't offer us a means to distinguish these objects.
- Apart from by using an intensional equality type.
- ITT doesn't identify isomorphic types.
- On the other hand it doesn't offer us a means to distinguish these objects.
- Apart from by using an intensional equality type.

Extensional Type Theory (ETT)

- In the sense of NUPRL.
- Explains Type Theory by a realizability semantics.
- Types are **collections** of untyped computational objects.
- The inhabitance relation is justified by a notion of external truths.
- Identifies definitional equality and equality type (equality reflection).
- Thus extensionally equal functions are identified.
- But isomorphic types are not.
- Indeed equality reflection forces proof-irrelevance.
- This makes it impossible to identify isomorphic types.
- Hence ETT is actually anti-extensional in this sense.

Homotopy Type Theory

- HoTT fixes both incompleteness issues of ITT by adopting functional extensionality and univalence.
- View types as weak infinity groupoids.
- Equality types expose this structure.
- J now reflects the infinity groupoid structure.
- Less clear what is the computational explanation.
  (cubical sets, . . .)
- Now reason why β-equality for J should hold definitionally.
- Relic from the inductive understanding of equality in ITT.

Two level theories

- Incompleteness of current HoTT: we cannot define semi-simplicial types internally.
- We introduce a strict equality in the sense of intensional type theory to avoid coherence problem.
- Pretypes vs. (fibrant) types
- Should the pretype of natural numbers be the same as the type.
- If yes, we can define a type of semisimplicial types
- Can we introduce an operation of fibrant replacement?
  What are its rules?
- Can have intensional and extensional view of the same objects in the same system.
- Different views of a car:

  extensional press gas pedal, it drives!

  intensional how does the engine works?
- Try to stick to the extensional view.

Structuralism?

- Category theory a good way to structure mathematical objects.
- Doesn't provide element-level construction.
- HoTT: perfect fit, construct types uniquely given by their universal properties.
- Desire to introduce higher structures gives rise to higher categorical constructions.

Levels of Mathematics

element level Mathematics

set level Mathematics

structural Mathematics

- Much work takes place on the 1st two levels.
- Set-level HoTT = set-truncated fragment of HoTT.
- Easier to model meta-theoretically: setoid model.

How to convince Mathematicians?

- Don't try!
- Can't teach an old dog new tricks.
- Concentrate on the young dogs.
- Formal Mathematics using interactive proof systems opportunity to change foundations.
- It would be a shame if the tools of tomorrow use the mathematics of yesterday.